Multi-Hamiltonian Structure of Lotka-Volterra and Quantum Volterra Models

Christofer Cronström[*)]

NORDITA
Blegdamsvej 17,
DK-2100 Copenhagen Ø, Denmark

and

Milan Noga[†)]

Department of Theoretical Physics, Comenius University
Mlynska Dolina, Bratislava, Slovak Republic

ABSTRACT

We consider evolution equations of the Lotka-Volterra type, and elucidate especially their formulation as canonical Hamiltonian systems. The general conditions under which these equations admit several conserved quantities (multi-Hamiltonians) are analysed. A special case, which is related to the Liouville model on a lattice, is considered in detail, both as a classical and as a quantal system.

[*)] Permanent address: Department of Theoretical Physics, University of Helsinki, Finland.
e-mail: cronstrom@ phcu.helsinki.fi.
[†)]e-mail: Milan.Noga@fmph.uniba.sk

1 Introduction

The Lotka-Volterra model [1, 2] is defined by a set of non-linear differential equations of the following form,

$$\frac{dw_A}{dt} = \epsilon_A w_A + \sum_{B=1}^{N} F_{AB} w_A w_B \ , \ A = 1, 2, ...N \tag{1}$$

In the equations above the quantities w_A, $A = 1, 2, ..., N$ are to be determined in terms of the given constant quantities ϵ_A and F_{AB}. The Lotka-Volterra equations are principally known as qualitative models in population dynamics with N interacting species, as well as rate equations for chemical reactions with N constituents. However, the equations in question have actually turned out to be rather universal with different applications in physics [3], as witnessed e.g. by the case of the Liouville model formulated on a lattice [4].

The matrix F in Eq.(1) ought to fulfill certain conditions which relate to the concept "crowding inhibits growth". This condition will here be related to the requirement that the matrix F be antisymmetric,

$$F_{AB} = -F_{BA} \tag{2}$$

We wish to elucidate the circumstances under which the equations (1) including the antisymmetry condition (2), admit a canonical Hamiltonian formulation. This question has been answered in a satisfactory manner by Kerner [5] already quite some time ago, however with the restriction to an *even* number N of species and with the condition that the matrix F be *regular*. Here we give a general analysis, valid for even or odd N, as well as for regular or singular matrices F. In the singular case (which always occurs if N is odd), the equations (1) admit additional conserved quantities besides the Hamiltonian, under certain circumstances. We analyse these circumstances in detail. In special cases this phenomenon has been noted in the literature (e.g. [6]), and has been referred to as a multi-Hamiltonian structure.

The analysis of the general Lotka-Volterra equations (1) including the condition (2) is given in Sec. II below; the next Sec. III then deals with a particular example corresponding to an odd number $N = 3$ of species.

In Sec. IV we consider a specialisation of the Lotka-Volterra model, which appeared in a formulation of the Liouville model on the lattice [4], the quantal formulation of which was further considered by Volkov [7] who introduced the name *"Quantum Volterra model"* in this connection.

The Hamiltonian proposed in [4] and in [7] is actually one of the additional conserved quantities that appear as a consequence of the special features of the model under consideration, and differs from the Hamiltonian derived in this paper. Our alternative formulation permits a simple canonical quantization of the model under consideration.

The final Sec. V contains a summary and discussion.

2 The General Lotka-Volterra Equation

In analysing Eq. (1) it is convenient to introduce new variables as follows,

$$\xi_A = \log w_A \tag{3}$$

Then Eq. (1) can be rewritten as follows,

$$\dot{\xi}_A = \epsilon_A + \sum_{B=1}^{N} F_{AB} \exp \xi_B \tag{4}$$

In order to analyse Eq. (1) further, it is convenient to refer to the socalled normal form of the antisymmetric matrix F. It is well known [8] that, by making an appropriate basis transformation, any antisymmetric $N \times N$ matrix F can be transformed into the following normal form:

$$\begin{pmatrix} 0 & -k_1 & \dots & \dots & \dots & \dots & \dots & \dots \\ +k_1 & 0 & \dots & \dots & \dots & \dots & \dots & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & 0 & -k_n & \dots & \dots & \dots \\ \dots & \dots & \dots & +k_n & 0 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \ddots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \end{pmatrix} \tag{5}$$

where the (positive) quantities $k_\alpha,\ \alpha = 1, ..., n$ are the square roots of those characteristic values of the matrix $-F^2$, which are different from zero. The number n of non-zero characteristic values k_α^2 is thus given by the rank ($2n$, say) of the matrix F.

It is furthermore convenient to relate the normal form (5) to the following linear equations,

$$\sum_{B=1}^{N} F_{AB} x_{\alpha B} = +\ k_\alpha y_{\alpha A},\ \alpha = 1, ..., n \tag{6}$$

$$\sum_{B=1}^{N} F_{AB} y_{\alpha B} = -\ k_\alpha x_{\alpha A},\ \alpha = 1, ..., n \tag{7}$$

and

$$\sum_{B=1}^{N} F_{AB} z_{\beta B} = 0,\ \beta = 1, ..., N - 2n \tag{8}$$

with the understanding that Eq. (8) is empty if the rank of the matrix F is N (so that $N = 2n$), in which case the matrix F is *regular*,

$$det\ F \neq 0. \tag{9}$$

The cases N even or odd differ qualitatively in general, since in the latter case the matrix F is *necessarily* singular. Then Eq. (8) has an odd number of nontrivial solutions. Eq. (8)

may of course also have non-trivial solutions if N is even, in which case there is necessarily an even number of such solutions.

We ortho-normalize the vectors $x_{\alpha A}$, $y_{\alpha A}$, and $z_{\beta A}$, properly,

$$(x_\alpha, x_\beta) = \delta_{\alpha\beta},\ (y_\alpha, y_\beta) = \delta_{\alpha\beta},\ (z_\alpha, z_\beta) = \delta_{\alpha\beta} \tag{10}$$

where the inner product (u, v) of any vectorlike quantities u_A and v_A is defined as follows,

$$(u, v) \equiv \sum_{A=1}^{N} u_A v_A \tag{11}$$

The equations (6), (7) and (8) imply the following orthogonality relations,

$$(x_\alpha, y_\beta) = 0,\ \alpha, \beta = 1, ..., n; \tag{12}$$

and

$$(x_\alpha, z_\beta) = (y_\alpha, z_\beta) = 0,\ \alpha = 1, ..., n, \beta = 1, ..., N - 2n \tag{13}$$

After these preliminaries we return to the equations (1). Contracting Eq. (1) with any solution z_β of Eq. (8), one obtains, in view of the antisymmetry condition (2),

$$(z_\beta, \dot{\xi}) = (z_\beta, \epsilon) \equiv r_\beta \tag{14}$$

where we use the inner product notation given above in Eq. (11). Thus,

$$(z_\beta, \xi) = r_\beta t + K_\beta\ ,\ \beta = 1, ..., N - 2n \tag{15}$$

where the quantities K_β are arbitrary constants of integration. The equations (15) define $N-2n$ *constraints* among the variables ξ_A; the number of these constraints is equal to the number of linearly independent solutions (if any) to the zero-eigenvalue equations (8). In particular, if

$$r_\beta \equiv (z_\beta, \epsilon) \neq 0\ \beta = 1, ..., N - 2n \tag{16}$$

then the $N - 2n$ quantities (z_β, ξ) are linear functions of t.

The equations (1) still contain $2n$ unknowns; one obtains a convenient set of $2n$ equations for these unknowns by contracting the equations (1) with the solutions x_α and y_α of the equations (6) and (7), respectively. Thus,

$$(x_\alpha, \dot{\xi}) = (x_\alpha, \epsilon) - \sum_{B=1}^{N} k_\alpha y_{\alpha B} \exp \xi_B \tag{17}$$

and

$$(y_\alpha, \dot{\xi}) = (y_\alpha, \epsilon) + \sum_{B=1}^{N} k_\alpha x_{\alpha B} \exp \xi_B \tag{18}$$

It should be noted that the equations (14), (17) and (18) are equivalent to the original equations (1) under the condition (2), since the orthonormal vectors x_α, y_α and z_α form a complete set in N dimensions. Thus,

$$\xi_A = \sum_{\alpha=1}^{n} [(x_\alpha, \xi) x_{\alpha A} + (y_\alpha, \xi) y_{\alpha A}] + \sum_{\beta=1}^{N-2n} (z_\beta, \xi) z_{\beta A} \tag{19}$$

The *even* number of equations (17) and (18) are in fact Hamiltonian equations in coordinates q_α and momenta p_α defined as follows,

$$p_\alpha \equiv C_\alpha (x_\alpha, \xi) \tag{20}$$

and

$$q_\alpha \equiv D_\alpha (y_\alpha, \xi) \tag{21}$$

where the quantities C_α and D_α are constants, which satisfy the following condition,

$$C_\alpha D_\alpha k_\alpha = 1 \tag{22}$$

Namely, let

$$H(p, q; t) \equiv \sum_{B=1}^{N} \exp\{\xi_B(p, q; t)\} - \sum_{\alpha=1}^{n} [C_\alpha (x_\alpha, \epsilon) q_\alpha - D_\alpha (y_\alpha, \epsilon) p_\alpha] \tag{23}$$

where the quantity $\xi(p, q; t)$ is expressed in terms of the quantities p_α, q_α defined above, as well as in terms of the quantities r_β and K_β defined in Eq. (14) and (15), respectively, according to Eq. (19),

$$\xi_A(p, q; t) = \sum_{\alpha=1}^{n} \left[(C_\alpha^{-1} p_\alpha) x_{\alpha A} + (D_\alpha^{-1} q_\alpha) y_{\alpha A} \right] + \sum_{\beta=1}^{N-2n} (r_\beta t + K_\beta) z_{\beta A} \tag{24}$$

Using Eq. (24) and Eq. (22) one obtains straightforwardly the following results from Eq. (23),

$$\frac{\partial H}{\partial p_\alpha} = D_\alpha \left[\sum_{B=1}^{N} k_\alpha x_{\alpha B} \exp\{\xi_B(p, q; t)\} + (y_\alpha, \epsilon) \right] \tag{25}$$

and

$$\frac{\partial H}{\partial q_\alpha} = C_\alpha \left[\sum_{B=1}^{N} k_\alpha y_{\alpha B} \exp\{\xi_B(p, q; t)\} - (x_\alpha, \epsilon) \right] \tag{26}$$

Then, using the relations just given, it is a simple matter to verify that Eq. (17) is nothing but the following,

$$\frac{dp_\alpha}{dt} = -\frac{\partial H}{\partial q_\alpha} \tag{27}$$

Likewise, Eq. (18) is equivalent to the following,

$$\frac{dq_\alpha}{dt} = +\frac{\partial H}{\partial p_\alpha} \tag{28}$$

It should be noted that the Hamiltonian (23) is in general *explicitly time-dependent* if the zero-eigenvalue equations (8) have non-trivial solutions. This requires that the matrix F ocurring in the Lotka-Volterra equations (1) be *singular*, which is always the case if the dimensionality N of the system is an odd integer. Be that as it may, as we have just demonstrated, there is in any case always an *even-dimensional* subset of the equations (1) which can be written in canonical Hamiltonian form, in terms of canonical momenta (20) and coordinates (21), with the expression (23) as a Hamiltonian.

The explicit time-dependence of the Hamiltonian (23) disappears if the vector ϵ (the array of rate-constants ϵ_A) occuring in Eq. (1) is orthogonal to all the solutions z_β of the zero-eigenvalue equation (8),

$$(z_\beta, \epsilon) = 0 \ , \ \beta = 1, ..., N - 2n \tag{29}$$

If the equations (29) are in force, we have, according to Eq. (15),

$$(z_\beta, \xi) = K_\beta \ , \ \beta = 1, ..., N - 2n \tag{30}$$

Each orthogonality condition (29) with a fixed value of the index β, gives rise to a *conserved* quantity, namely the corresponding expression in Eq. (30), according to Eqns. (15).

We conclude this section by summarizing the results obtained so far:

The Lotka-Volterra equations (1), including the condition (2), can always be written as a system of canonical Hamiltonian equations in an even number of unconstrained canonical variables, together with a set of explicitly solvable constraints, which are time-dependent in general. This set is non-empty, i.e. the constraints in question occur in general, if the matrix F in Eq. (1) is singular. The number of constraints equals the number of linearly independent eigen-vectors z_β of the matrix F, which correspond to zero-eigenvalues.

For non-singular matrices F the Hamiltonian does not depend on time explicitly. In the singular case, the Hamiltonian is in general explicitly time-dependent. This time-dependence disappears from the Hamiltonian, if all the eigen-vectors z_β are orthogonal to the vector $(\epsilon_1, ..., \epsilon_N)$ made up by the rate-constants ϵ_A in the basic equations (1). Each orthogonality condition of the aforementioned kind gives rise to a (time-independent) linear constraint among the variables ξ_A in the equations (4), which are equivalent to the basic equations (1).

3 An Example Involving Three Degrees of Freedom

We consider below a system of the Lotka-Volterra type, with three degrees of freedom, which has been analysed in great detail by Grammaticos et.al [9], with the objective of making a systematic search for first integrals of the system in question. A special case of this system has been presented by Nutku [6] as an example of a bi-Hamiltonian system of the Lotka-Volterra type.

The general system is the follwing (a, b, c and λ, μ, ν are constants),

$$\dot{s} = s(\lambda + ct + u) \ , \dot{t} = t(\mu + s + au) \ , \dot{u} = u(\nu + bs + t) \tag{31}$$

Rescaling the variables in Eq. (31) as follows,

$$s \rightarrow \frac{1}{\alpha_1} s \equiv w_1 \ , t \rightarrow \frac{1}{\alpha_2} t \equiv w_2 \ u \rightarrow \frac{1}{\alpha_3} u \equiv w_3 \tag{32}$$

where the parameters $\alpha_n, n = 1, 2, 3$, are at our disposal, and using the ξ-variables introduced in (3), we get an equation of the form (4) from Eq. (31),

$$\dot{\xi}_A = \epsilon_A + \sum_{B=1}^{3} F_{AB} \exp \xi_B \tag{33}$$

where,

$$\epsilon_1 = \lambda \ , \ \epsilon_2 = \mu \ , \ \epsilon_3 = \nu \tag{34}$$

and

$$F = \begin{pmatrix} 0 & \alpha_2 c & \alpha_3 \\ \alpha_1 & 0 & \alpha_3 a \\ \alpha_1 b & \alpha_2 & 0 \end{pmatrix} \tag{35}$$

The matrix F defined by Eq. (35) does not fulfill the antisymmetry condition (2) as such, but becomes antisymmetric under the follwing conditions,

$$\alpha_1 = -\alpha_2 c \ , \alpha_2 = -\alpha_3 a \ , \alpha_3 = -\alpha_1 b \tag{36}$$

But Eqs. (36) can be fulfilled if and only if

$$abc = -1 \tag{37}$$

From now on, we *assume* that the parameters a, b, c satisfy the condition (37). Without essential loss of generality, we can choose one parameter α_n at will in the equations above. Taking $\alpha_1 = 1$, we satisfy the equations (36) as follows,

$$\alpha_1 = 1, \ \alpha_2 = ab, \ \alpha_3 = -\, b \tag{38}$$

We then apply the results of Sec. II to the system of equations above.

In the first place we have to consider the following system of linear (eigenvalue) equations (compare (6), (7) and (8)),

$$\sum_{B=1}^{3} F_{AB} x_B = +\, k y_A \ , \ \sum_{B=1}^{3} F_{AB} y_B = -\, k x_A \ , \ \sum_{B=1}^{3} F_{AB} z_B = 0 \tag{39}$$

The following are appropriately orthonormalised solutions to these equations,

$$x = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{1+b^2}} \\ \frac{b}{\sqrt{1+b^2}} \end{pmatrix} \ , \ y = \frac{1}{k} \begin{pmatrix} -\sqrt{1+b^2} \\ -\frac{ab^2}{\sqrt{1+b^2}} \\ \frac{ab}{\sqrt{1+b^2}} \end{pmatrix} \ , \ z = \frac{1}{k} \begin{pmatrix} ab \\ -b \\ 1 \end{pmatrix} \tag{40}$$

with

$$k = \sqrt{1 + b^2 + a^2 b^2} \tag{41}$$

Using Eqns. (40) one now obtains the follwing constraint, according to Eq. (15),

$$(z, \xi) \equiv \frac{1}{k} [ab\xi_1 - b\xi_2 + \xi_3] = r_1 t + K_1 \tag{42}$$

where

$$r_1 \equiv (z, \epsilon) = \frac{1}{k} [ab\lambda - b\mu + \nu] \tag{43}$$

The remaining variables can then be taken to be the canonical variables defined in Eqns. (20, 21), i.e. in the present case,

$$p \equiv C(x, \xi) = \frac{C}{\sqrt{1+b^2}} (\xi_2 + b\xi_3) \tag{44}$$

and

$$q \equiv D(y, \xi) = -\, \frac{D}{k\sqrt{1+b^2}} ((1+b^2)\xi_1 + ab^2\xi_2 - ab\xi_3) \tag{45}$$

where the constants C and D are related as follows

$$CDk = 1 \tag{46}$$

but are otherwise arbitrary.

The equations (42), (44) and (45) can easily be inverted,

$$\begin{aligned} \xi_1 &= \frac{ab}{k}(r_1 t + K_1) - \sqrt{1+b^2}\, Cq \\ \xi_2 &= -\frac{b}{k}(r_1 t + K_1) + \frac{k}{\sqrt{1+b^2}} Dp - \frac{ab^2}{\sqrt{1+b^2}} Cq \\ \xi_3 &= \frac{1}{k}(r_1 t + K_1) + \frac{bk}{\sqrt{1+b^2}} Dp + \frac{ab}{\sqrt{1+b^2}} Cq \end{aligned} \tag{47}$$

Using the results above, it is simple to verify that the variables p and q defined in the equations (44), (45) above, satisfy canonical Hamiltonian equations, i.e. the equations (27), (28), with the following Hamiltonian H,

$$H = \frac{1}{k^2}\left[(\mu + b\nu)\xi_1 - (\lambda - ab\nu)\xi_2 - b(\lambda + a\mu)\xi_3\right] + \sum_{A=1}^{3} \exp \xi_A \tag{48}$$

The Hamiltonian (48) is in general explicitly *time dependent*, since the sum of exponentials in (48) above, depends on time in general, as can be inferred from the equations (42). However, if the orthogonality condition (29) is in force, i.e. if

$$r_1 \equiv (z, \epsilon) = \frac{1}{k}[ab\lambda - b\mu + \nu] = 0 \tag{49}$$

then the Hamiltonian (48) becomes a constant of motion. However, if the condition (49) is in force, then there exists a *second* constant of motion, according to Eq.(42), namely the following,

$$\frac{1}{k}[ab\xi_1 - b\xi_2 + \xi_3] \tag{50}$$

The appearance of the two conserved quantities (48) and (50) under the condition (49) is just the phenomenon which has been referred to by the term bi-Hamiltonian in the present context by Nutku [6].

This terminology is perhaps a little unfortunate, despite the fact that the quantity (50) can be understood as a Hamiltonian for the system (33) (if condition (49) is in force) in a certain general sense [10]. On the contrary, the Hamiltonian (48), is a Hamiltonian in a strict sense, regardless of whether the condition (49) is true or not. By strict sense is here meant that the formulation involving the Hamiltonian (48) is explicitly canonical, involving a known pair of canonical variables p and q. So, in the general case, only the quantity (48) merits a designation as a Hamiltonian in the strict canonical sense employed in this paper.

The considerations above are naturally generalisable to systems with more than three degrees of freedom, in view of the general results given in Sec. II.

4 The Liouville Model on a Lattice

The Liouville model on a lattice discussed by Faddeev and Takhtajan [4] and by Volkov [7] is specified by the following classical equations of motion, which is a special system of Lotka-Volterra equations,

$$\frac{dw_A}{dt} = \frac{1}{2\Delta} w_A(w_{A+1} - w_{A-1}) \ , \ A = 1, ..., N \tag{51}$$

where N is an *even* positive integer, and where Δ is a parameter. The variables w_A are furthermore supposed to satisfy the following periodicity condition,

$$w_{N+n} = w_n \ , \ n = 0, 1, 2, ... \tag{52}$$

Comparing Eqs. (51) with the general Lotka-Volterra equations (1) one observes that the model defined above by the Eqns. (51) corresponds to the case of vanishing rate constants ϵ_A,

$$\epsilon_A = 0 \ , \ A = 1, 2, ..., N \tag{53}$$

and to the case in which the matrix F has the following form,

$$(F_{AB}) = \begin{pmatrix} 0 & \lambda & \dots & \dots & \dots & \dots & \dots & -\lambda \\ -\lambda & 0 & \lambda & \dots & \dots & \dots & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \dots & \dots & -\lambda & 0 & \lambda & \dots & \dots & \dots \\ \dots & \dots & \dots & -\lambda & 0 & \lambda & \dots & \dots \\ \dots & \dots & \dots & \dots & \ddots & \ddots & \ddots & \dots \\ \dots & \dots & \dots & \dots & \dots & -\lambda & 0 & \lambda \\ \lambda & \dots & \dots & \dots & \dots & \dots & -\lambda & 0 \end{pmatrix} \tag{54}$$

where

$$\lambda \equiv \frac{1}{2\Delta} \tag{55}$$

The matrix (54) is *singular*,

$$det \ F = 0 \tag{56}$$

This circumstance together with the conditions (53) implies the existence of additional conserved quantities (constants of motion) besides the Hamiltonian, according to the general results presented in Sec. II.

At first we give a brief analysis of the canonical formulation for the classical latticised Liouville model, along the lines presented in Sec. II.

4.1 Canonical Hamiltonian Formulation of the Classical Liouville Model on a Lattice

First we rewrite Eq. (51) using the variable (3),

$$\dot{\xi}_A = \lambda \left(\exp \xi_{A+1} - \exp \xi_{A-1} \right) \equiv \sum_{B=1}^{N} F_{AB} \exp \xi_B \tag{57}$$

where the matrix F given by the expression (54). According to the general method developed in Sec. II, we consider to begin with the eigenvalue-equation (8), i.e. the following equation,

$$\sum_{B=1}^{N} F_{AB} z_{\beta,B} = 0 \tag{58}$$

As is immediately verified, Eq. (58) has two linearly independent (normalized) solutions z_β, which we take to be the following,

$$z_{1,A} = \frac{1}{\sqrt{2N}} (1 - (-1)^A) \ , \ A = 1, ..., N \tag{59}$$

and

$$z_{2,A} = \frac{1}{\sqrt{2N}} (1 + (-1)^A) \ , \ A = 1, ..., N \tag{60}$$

The existence of the two non-trivial solutions (59) and (60) implies the existence of the following two conserved quantities, according to Eqs. (29) and (30),

$$\sqrt{\frac{2}{N}} \sum_{A=1}^{\frac{1}{2}N} \xi_{2A-1} = K_1 \tag{61}$$

and

$$\sqrt{\frac{2}{N}} \sum_{A=1}^{\frac{1}{2}N} \xi_{2A} = K_2 \tag{62}$$

Needless to say, the Hamiltonian (23), which now takes the following simple form, is also conserved,

$$H = \sum_{B=1}^{N} \exp \xi_B \equiv \sum_{B=1}^{N} w_B \tag{63}$$

We then have to consider the equations (6) and (7) with the matrix F given by (54).

After calculations one finds the follwing solutions $x_{\alpha A}$ and $y_{\alpha A}$,

$$x_{\alpha A} = \frac{1}{\sqrt{2N}} (1 + (-1)^A) \left[\cos(A \varphi_\alpha) + \sin(A \varphi_\alpha) \right] \ , \ \alpha = 1, 2, ..., \frac{1}{2} N - 1. \tag{64}$$

and

$$y_{\alpha A}=\frac{1}{\sqrt{2N}}(1-(-1)^{A})\left[\cos(A\varphi_{\alpha})-\sin(A\varphi_{\alpha})\right]\ ,\ \alpha=1,2,...,\frac{1}{2}N-1. \tag{65}$$

where

$$\varphi_{\alpha}=\frac{2\pi\alpha}{N}\ ,\ \alpha=1,2,...,\frac{1}{2}N-1. \tag{66}$$

The eigenvalue parameter k_α is expressed in terms of φ_α as follows,

$$k_{\alpha}=2\lambda\sin\varphi_{\alpha}\ ,\ \alpha=1,2,...,\frac{1}{2}N-1. \tag{67}$$

Applying the general formulae (20) and (21) we now obtain the following expressions for the canonical momenta and coordinates, respectively,

$$p_{\alpha}=\frac{2C_{\alpha}}{\sqrt{N}}\sum_{A=1}^{\frac{1}{2}N}\xi_{2A}\sin[2A\varphi_{\alpha}+\frac{\pi}{4}] \tag{68}$$

and

$$q_{\alpha}=\frac{2D_{\alpha}}{\sqrt{N}}\sum_{A=1}^{\frac{1}{2}N}\xi_{2A-1}\cos[(2A-1)\varphi_{\alpha}+\frac{\pi}{4}] \tag{69}$$

The inverses of Eqns. (68), (69) are obtained straightforwardly, either by using the general result (24) or by evaluating the appropriate sums involving trigonometric functions,

$$\xi_{A}=\sum_{\alpha}C_{\alpha}^{-1}p_{\alpha}x_{\alpha A}+\sum_{\alpha}D_{\alpha}^{-1}q_{\alpha}y_{\alpha A}+\sum_{\beta=1}^{2}K_{\beta}z_{\beta,A} \tag{70}$$

where the constants K_β are given in Eqns. (61), (62).

We conclude this sub-section by giving a Poisson-bracket formulation of the equations (57). The basic Poisson-bracket is the following

$$\{\xi_{A},\xi_{B}\}_{PB}\equiv\sum_{\alpha=1}^{\frac{1}{2}N-1}\left[\frac{\partial\xi_{A}}{\partial q_{\alpha}}\frac{\partial\xi_{B}}{\partial p_{\alpha}}-\frac{\partial\xi_{B}}{\partial q_{\alpha}}\frac{\partial\xi_{A}}{\partial p_{\alpha}}\right]=-\sum_{\alpha=1}^{n}k_{\alpha}\left[x_{\alpha A}y_{\alpha B}-y_{\alpha A}x_{\alpha B}\right] \tag{71}$$

Finally, evaluating the relevant trigonometric sums in (71) (compare Eqns. (64), (65)), one obtains,

$$\{\xi_{A},\xi_{B}\}_{PB}=\lambda\left(\delta_{A,B-1}-\delta_{A,B+1}\right)\equiv F_{AB} \tag{72}$$

Using the result (72) it is straightforward to verify that the equations (57) are reproduced by the following expressions,

$$\dot{\xi}_{A}=\{\xi_{A},H\}_{PB}=\sum_{C=1}^{N}\{\xi_{A},\xi_{C}\}_{PB}\frac{\partial H}{\partial\xi_{C}} \tag{73}$$

where the Hamiltonian is given by Eq. (63)

Reverting to the original variables w_A (compare Eq. (3)), one gets the following Poisson-bracket relations from (72),

$$\{w_A, w_B\}_{PB} = \lambda w_A w_B \left(\delta_{A,B-1} - \delta_{A,B+1}\right) \tag{74}$$

It is perhaps necessary to emphasize that the result (74) is firmly based on the canonical structure derived above for the latticised Liouville model, and not merely an example of a bracket structure which is consistent with the Hamiltonian (63) and the equations of motion (51).

In the aforementioned paper by Volkov [7], an alternative Hamiltonian formalism to the one presented above has been proposed for the quantization of the system defined by the equations (51).

The Hamiltonian H_V used by Volkov is simply the sum of the two constants of motion (61) and (62), apart from a multiplicative constant,

$$H_V = -\frac{1}{2\gamma\Delta} \sum_{A=1}^{N} \log w_A \tag{75}$$

where γ is an arbitrary coupling constant, which is introduced for convenience. The quantity (75) is a constant of motion, since it is the sum of two constants of motion (compare Eqns. (61, 62)).

The Hamiltonian scheme of Volkov consists of a bracket formulation of the equations (51) with (75) as a Hamiltonian. Thus,

$$\dot{w}_A = \{w_A, H_V\} = \sum_{B=1}^{N} \{w_A, w_B\} \frac{\partial H_V}{\partial w_B} \tag{76}$$

The basic bracket given by Volkov is the following (after correction of a crucial printing error; an overall factor $w_A w_B$ is missing from Volkovs expression, Eq. (23) of Ref. [7]),

$$\{w_A, w_B\} = \frac{1}{2}\gamma w_A w_B \left[(4 - w_A - w_B)(\delta_{A+1,B} - \delta_{A-1,B}) + w_{A-1}\delta_{A-2,B} - w_{A+1}\delta_{A+2,B}\right] \tag{77}$$

It is indeed simple to verify, that the equations (51) are reproduced as bracket equations (76) if one uses the bracket (77). However, it should be noted that the Hamiltonian H_V defined in Eq. (75) which is used in this context, is not bounded from below, and does therefore not permit the construction of classical statistical mechanics for the (classical) system in question. On the other hand, the Hamiltonian H defined in Eq. (63), which has been obtained here on the basis of general considerations of the canonical structure of systems of the Lotka-Volterra type, *is* bounded from below, and is thus preferable also from this point of view.

4.2 The Canonical Quantization of the Liouville Model on a Lattice

The quantization of the model considered above is straightforward if one uses the canonical structure derived here as a starting point. Following the usual prescription of replacing Poisson brackets involving classical canonical variables by commutators involving the corresponding operators one has the following canonical quantum operator relations,

$$[\hat{p}_\alpha, \hat{p}_\beta] = 0,\ [\hat{q}_\alpha, \hat{q}_\beta] = 0,\ [\hat{q}_\alpha, \hat{p}_\beta] = i\hbar\delta_{\alpha\beta} \tag{78}$$

From the canonical commutators (78) follows indeed the quantization prescription given below in terms of the basic variables ξ of the problem,

$$\{\xi_A, \xi_B\}_{PB} \longrightarrow \frac{1}{i\hbar}\left[\hat{\xi}_A, \hat{\xi}_B\right] \tag{79}$$

where quantal variables are distinguished by a hat, thus: $\hat{\xi}$.

The Hamiltonian H given in Eq. (63) can immediately be generalized to an operator Hamiltonian $\hat{H}$. Using the inverse relations (70), or more explicitly the following relations,

$$\hat{\xi}_{2A} = \frac{4\lambda}{\sqrt{N}} \sum_{\alpha=1}^{\frac{1}{2}N-1} D_\alpha \hat{p}_\alpha \sin\varphi_\alpha \sin[(2A)\varphi_\alpha + \frac{\pi}{4}] + \sqrt{\frac{2}{N}} K_2 \tag{80}$$

and

$$\hat{\xi}_{2A-1} = \frac{4\lambda}{\sqrt{N}} \sum_{\alpha=1}^{\frac{1}{2}N-1} C_\alpha \hat{q}_\alpha \sin\varphi_\alpha \cos[(2A-1)\varphi_\alpha + \frac{\pi}{4}] + \sqrt{\frac{2}{N}} K_1 \tag{81}$$

one obtains the following expression,

$$\hat{H} = T(\hat{p}) + V(\hat{q}) \tag{82}$$

where

$$T(\hat{p}) = \sum_{A=1}^{\frac{N}{2}} \exp\left\{\sqrt{\frac{2}{N}} K_2 + \frac{4\lambda}{\sqrt{N}} \sum_{\alpha=1}^{\frac{1}{2}N-1} D_\alpha \hat{p}_\alpha \sin\varphi_\alpha \sin[(2A)\varphi_\alpha + \frac{\pi}{4}]\right\} \tag{83}$$

and

$$V(\hat{q}) = \sum_{A=1}^{\frac{N}{2}} \exp\left\{\sqrt{\frac{2}{N}} K_1 + \frac{4\lambda}{\sqrt{N}} \sum_{\alpha=1}^{\frac{1}{2}N-1} C_\alpha \hat{q}_\alpha \sin\varphi_\alpha \cos[(2A-1)\varphi_\alpha + \frac{\pi}{4}]\right\} \tag{84}$$

The Hamiltonian (82) is expressed as a sum of terms depending separately on the momentum variables $\hat{p}$ and coordinate variables $\hat{q}$, respectively, so there are no essential quantum ordering problems in the quantization procedure outlined above.

The quantization procedure considered by Volkov [7] is not explicitly canonical and can therefore not be compared directly with the canonical standard procedure given above.

5 Summary and Discussion

In this paper we have given a general analysis of the canonical structure of the system of differential equations which are known as the Lotka-Volterra model, supplemented with an antisymmetry condition (which is crucial for our analysis) which is also frequently assumed in connection with the Lotka-Volterra equations. Altogether this defines a dynamical system, with a finite number of degrees of freedom, which we call the Lotka-Volterra system.

It has been shown that the Lotka-Volterra system always can be resolveded into an explicitly canonical Hamiltonian subsytem, involving an even number of canonical equations, together with a set of explicitly solvable constraints, which are time-dependent in general. The set of constraints is empty if the matrix which defines the interaction between the various components in the model, is regular. A set of canonical variables (pairs of coordinates and momenta) is explicitly constructed for the Lotka-Volterra system.

Furthermore, it has been shown that the Lotka-Volterra model gives rise to several conserved quantities (constants of motion) if the parameters of the model satisfy certain conditions, which have a general geometric characterisation as certain orthogonality conditions. The occurrence of such additional constants of motion has been termed a "multi-Hamiltonian structure".

As an illustration, a Lotka-Volterra system involving three degrees of freedom has been considered in detail.

An important special Lotka-Volterra system, which is associated with the Liouville model on a lattice has also been analysed in detail. The canonical structure and Hamiltonian formulation of the system in question has been shown to be a straightforward consequence of the general formalism developed in this paper. The canonical quantization for this case, which follows straightforwardly from the canonical structure of the underlying classical model, has been outlined in detail. This canonical quantization procedure differs in a non-trivial way from a previously proposed Hamiltonian procedure [4, 7] for the model under consideration.

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